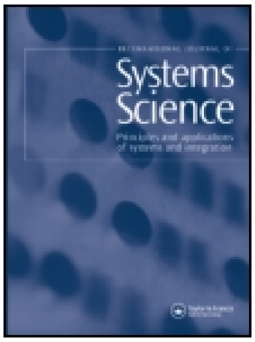

International Journal of Systems Science

ISSN: 0020-7721 (Print) 1464-5319 (Online) Journal homepage: https://www.tandfonline.com/loi/tsys20

A frequency-domain approach to bifurcations in control systems with saturation

J. Aracil , E. Ponce & E. Ponce

To cite this article: J. Aracil , E. Ponce & E. Ponce (2000) A frequency-domain approach to bifurcations in control systems with saturation, International Journal of Systems Science, 31:10, 1261-1271, DOI: 10.1080/002077200050165762

To link to this article: https://doi.org/10.1080/002077200050165762

Published online: 26 Nov 2010.

International Journal of Systems Science, 2000, volume 31, number 10, pages 1261–1271

A frequency-domain approach to bifurcations in control systems with saturation

J. ARACIL†, E. PONCE‡ and T. ÁLAMO‡

Bifurcation analysis of nonlinear control systems supplies a global perspective of the system behaviour modes. In this paper, it is shown that this analysis can be implemented with very elementary and classical tools, such as frequency-domain graphical methods. The methodology proposed is illustrated analysing the effects of a saturation nonlinearity on a system designed with a linear control law. The full morphology of qualitative behaviours and of polar plots for two-dimensional systems is displayed, pointing out how bifurcations give rise to the boundaries between regions with different qualitative behaviour modes. Also a case of a three-dimensional system is analysed.

1. Introduction

In some control applications, there naturally arises the question of studying the effects of saturation on the global behaviour of the system while it remains locally stable around the equilibrium point (the operating point). This kind of situation exemplifies the effects of nonlinearities on systems designed with linear methods. Control systems with nonlinearities give rise to nonlinear dynamical systems. The dynamical behaviour of such systems is much richer and more complicated than that of linear systems. Nonlinear systems display two main differences from the linear systems.

(i) They can show multiple steady regimes, and not only the point attractor associated with the operating point, as happens in linear systems.

(ii) They can exhibit long-term behaviours that are more complex than point attractors, such as limit cycles and chaotic attractors.

The search for these more complex behaviours is the goal of the qualitative analysis of nonlinear systems, and especially of bifurcation theory (Guckenheimer and Holmes 1983, Hale and Koçak 1991, Kuznetsov 1995).

Received 5 November 1998. Revised 21 September 1999. Accepted 5 October 1999.

† Escuela Superior de Ingenieros, Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain. e-mail: aracil@cartuja.us.es.

‡ Escuela Superior de Ingenieros, Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain.

Applications of the bifurcation theory are now very common in all branches of engineering (Nayfeh and Balachandran 1995, Strogatz 1995). These analysis techniques are becoming relevant for nonlinear control systems (Abed *et al.* 1996, Aracil *et al.* 1998a). With these tools, the system behaviour can be analysed not only considering the neighbourhood around the equilibrium point, as is traditionally done with linear methods, but also taking into account the whole state space.

Problems related to bifurcations in control systems have attracted attention for a long time (Mehra *et al.* 1977). Recently, the bifurcation analysis of nonlinear control systems has become more widespread. However, the applications appearing in scientific literature normally use more sophisticated mathematical approaches than those control specialists are used to. For instance, in a previous paper, Ponce *et al.* (1996) applied the results of Llibre and Ponce (1996), and Llibre and Sotomayor (1996) to the analysis of bifurcations in the class of linear control systems where saturation has been added. The results of these papers help to classify the system different behaviour modes. In particular, it is shown that bifurcations at infinity can give rise to an unstable limit cycle or to a pair of saddle points that state the limited attraction basin of the resulting nonlinear system. However, these results were presented in a rather mathematical language that is not commonly used by the control specialist. In this paper, the same kind of analysis and results are developed in the frequency domain, which is much more familiar to the designers of control systems.

The frequency response of nonlinear feedback control systems has usually been investigated with the describing function method (Mees 1981, Vidyasagar 1993, Khalil 1996). Under dissipativeness assumptions, some oscillation criteria also lying on frequency methods have been given by Leonov *et al.* (1996). Working in the frequency domain also has the advantage of its graphical character that gives a geometrical insight to the results, which is most appreciated by many systems designers. Applications of describing function methods to the bifurcation analysis of nonlinear control systems have already been reported (Mees 1981, Fukuma *et al.* 1984, Moiola and Chen 1993, 1996). Similar results to those developed here can be found in the work of Suárez *et al.* (1996). The methodology here employed is related to that in the papers by Genesio and Tesi (1992), Tesi *et al.* (1996), Alvarez and Curiel (1997), Alvarez *et al.* (1997) and Basso *et al.* (1997). However, the papers by Alvarez and Curiel and by Alvarez *et al.* deal mostly with detecting local bifurcations or with chaotic behaviour, whereas here the emphasis is on the global behaviour modes displayed by closed-loop, locally stable systems (those of practical interest), and especially how the bifurcations giving rise to the boundaries between regions of different qualitative behaviour can be detected using only frequency-domain transfer functions. In this way a classification of these behaviours and a global perspective on them can be reached.

In this paper a concrete case of nonlinear system is analysed: the control system formed by a linear system with an actuator that saturates. It can be argued that the results reported here have practical interest only for low-order systems. However, the practitioner engineer is used to work with reduced order models (the dominant poles), that are mostly second- or third-order systems. Also, saturation is an almost universal characteristic of practical actuators. So interest in the case is enough to deserve attention. Moreover, the methodology employed in the analysis is easily generalized to cover a wider class of systems.

On the other hand, another well-known result (Tarbouriech and Burgat 1997) that is also emphasized in this paper is the fact that unstable plants with saturating actuators give rise to closed-loop systems that are only locally stable. This is a particular case of the most general case of controlling unstable systems, which is a crucial point in control systems engineering. This requires respect for the unstable, as claimed by Stein (1989).

The remainder of this paper is organized as follows. In the next section, the structure of a linear control system with saturation in the feedback path is stated. Then the qualitative analysis of the system is developed, studying the equilibrium points and the limit cycles that the system can display. The study of these different qualitative state portraits is related to the bifurcations that state the boundaries between the different behaviour regions. Then a section is included where the full morphology of polar plots for two-dimensional systems is described, also showing the bifurcations associated with the transitions between regions of different qualitative behaviour. A section follows where a three-dimensional system is analysed. The paper is closed with conclusions.

2. Qualitative analysis in the frequency domain of stable closed-loop control systems with saturation

Consider the single input linear system in $\mathbb{R}^n$:

$$\dot{x} = Ax + Bu,$$

where $u \in \mathbb{R}$, and (A, B) is controllable, with a linear control law $u = -Kx$, which gives rise to a stable closed-loop linear control system with the state description

$$\dot{x} = (A - BK)x.$$

The closed-loop system is equivalent to a system with transfer function $G(s) = K(sI - A)^{-1}B$ with unity feedback. In the realm of linear control systems theory, there are many methods to obtain K in order to guarantee the stability of the operating point, assuming that the pair (A, B) is controllable. In the linear case, this stability is global.

However, the addition of nonlinearities (saturation, dead zone, etc.) can dramatically modify this situation. Suppose that a nonlinearity is included in the return chain, as shown in figure 1, so that the new state description is

$$\dot{x} = Ax + Bu = Ax - B\varphi(Kx), \qquad (1)$$

where $\varphi(.)$ stands for the nonlinearity characteristic. Although a more general case can be considered, in the following it will be assumed that $\varphi(.)$ is an odd function ($\varphi(y) = -\varphi(-y)$) that saturates in a monotonic way, that is

$$\lim_{y\to\infty} \left(\frac{\varphi(y)}{y}\right) = 0,$$

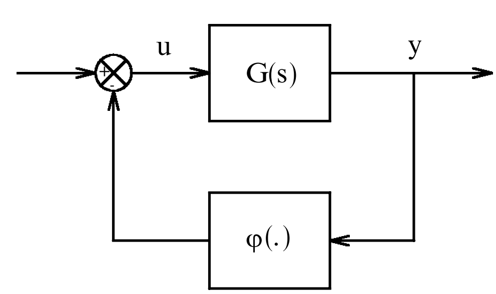

Figure 1. System with a nonlinearity in the feedback path.

with $\varphi(y)/y$ decreasing for $y > 0$. Without loss of generality, the slope of the nonlinearity at $y = 0$ will be supposed to be equal to one, that is $\varphi'(0) = 1$. The problem at stake is to analyse how the addition of the nonlinearity influences the global behaviour of the closed-loop system. The nonlinearity, even if it does not affect the stability of the operating point, can produce the appearance of new phenomena that do not occur in the linear model (equilibria different from the origin, limit cycles, etc.).

The global behaviour of the control system in figure 1, when the nonlinearity is the normalized saturation ($\varphi(x) = \operatorname{sgn}(x) \min\{|x|, 1\}$), can be studied with the help of the results of previously mentioned papers (Llibre and Ponce 1996, Llibre and Sotomayor 1996). Some of the results of these papers will be restated here using for the most part frequency-domain tools. The approach adopted in this paper differs from other approaches that also use frequency-domain tools (Genesio and Tesi 1992, Tesi *et al.* 1996, Alvarez and Curiel 1997, Alvarez *et al.* 1997, Basso *et al.* 1997) in that here we are not looking for all possible pathologies but emphasize the classification of global behaviours to be expected in locally stable nonlinear control systems.

2.1. *Equilibrium points analysis*

The first point is to realize that, with the addition of the odd and saturated nonlinearity φ, the system becomes nonlinear and may display several attractors. Therefore, the search for equilibria other than the origin should be undertaken.

The system equilibria are the points x^e such that $\dot{x} = 0$. From (1) one obtains

$$Ax^e = B\varphi(Kx^e). \tag{2}$$

Clearly, the origin is always a solution of (2) and its linearization matrix remains equal to $A - BK$. In checking other possible solutions of (2) with $x \neq 0$, consider first the case when $\det A = 0$. If $Kx^e \neq 0$, then it is deduced that $B \in \text{image } A$, which is in contradiction with the controllability of the system and the supposition of $\det A = 0$. It can be concluded that, if $x^e \neq 0$ is an equilibrium point when $\det A = 0$, then both $Ax^e = 0$ and $Kx^e = 0$ hold, which would lead to $(A - BK)x^e = 0$. This possibility will be discarded, because then $\det(A - BK) = 0$ and so it does not correspond to a stable closed-loop control design.

Assuming now that $\det A \neq 0$, from (2) it follows that

$$x^e = A^{-1}B\varphi(Kx^e),$$

and consequently

$$Kx^e = KA^{-1}B\varphi(Kx^e) = -G(0)\varphi(Kx^e),$$

where $G(0)$ is the static gain of the open-loop system. As a result, the equilibria correspond to the output values $y^e = Kx^e$ that are solutions of the scalar equation

$$y = -G(0)\varphi(y). \tag{3}$$

This equation can be solved graphically, as shown in figure 2. The points E_1, E_2 and O where the straight line $-y/G(0)$ intersects the curve $\varphi(y)$ correspond to different equilibria. The slope of the straight line is $\mu = -1/G(0)$ and so it cuts the graph of $\varphi(y)$ at points other than the origin only if $0 < \mu < 1$ ($-\infty < G(0) < -1$). When, for instance, the nonlinearity is the normalized saturation and $G(0) < -1$, there will be three equilibria: the origin O and two points E_1 and E_2 given by $\varphi(y^e) = \pm 1$; that is

$$x^e = \pm A^{-1}B;$$

for these equilibria it is guaranteed that $|y^e| = |Kx^e| = |G(0)| > 1$.

By moving μ, one can pass from one to three equilibria; this phenomenon is called a pitchfork bifurcation. In general, for saturated nonlinearities, the following pitchfork bifurcations must be taken into account.

(i) $G(0) = -1$. In this case, the straight line $-y/G(0)$ is tangent to the nonlinearity at $y = 0$ which results in a pitchfork bifurcation P_0 at the origin. By using the identity

$$\det(I - A^{-1}BK) = 1 - KA^{-1}B = 1 + G(0),$$

it is easy to see that

$$\det(A - BK) = \det[A(I - A^{-1}BK)] = [1 + G(0)] \det A.$$

Thus, if the system undergoes the pitchfork bifurcation at $G(0) = -1$ then a change of the stability of the origin is produced.

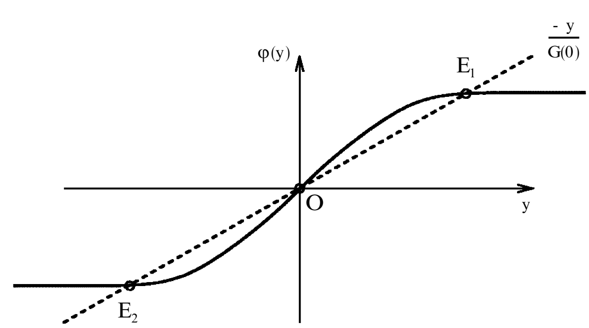

Figure 2. Graphical resolution of the equilibrium equation.

(ii) $G(0) = -\infty$. In this case, the straight line $-y/G(0)$ is horizontal. Then the assumption that

$$\lim_{y\to\infty}\left(\frac{\varphi(y)}{y}\right) = 0$$

monotonically assures the existence of two additional equilibria for any higher value of $G(0)$, provided that $G(0) > -1$. These two equilibria approach the origin as $G(0)$ grows, which may yield a reduction in the attraction basin at the origin. The appearance of these two equilibria from the infinity will be called a pitchfork bifurcation P_∞ at infinity and this phenomenon does not require any stability change for the origin.

2.2. *Predicting limit cycles*

Once the equilibrium points have been analysed, the problem is to search for more complex attractors, as are the limit cycles. The harmonic balance method can be used for this search. Assume that the system in figure 1 has a limit cycle of frequency ω. Then the signal y can be expressed by the Fourier expansion $y(t) = \sum_{-\infty}^{\infty} a_k\, e^{jk\omega t}$, and the output of the nonlinearity by $\varphi(y) = \sum_{-\infty}^{\infty} n_k\, e^{jk\omega t}$. From these expansions the harmonic balance equations are obtained (Mees 1981, chapter 5):

$$a_k + G(jk\omega)n_k = 0, \quad k = 0, \pm 1, \pm 2, \ldots. \qquad (4)$$

In most control applications with odd symmetry, a reasonable approximation is reached considering only the first order harmonic balance ($k = 1$). It should be recalled that the linear part $G(j\omega)$ must be a low-pass filter. In this case, to predict limit cycles the following equation must be solved:

$$1 + N(a)G(j\omega) = 0, \qquad (5)$$

where $N(a) = n_1/a_1$, $a_1 = a$, is the describing function for the nonlinearity. It is well known that for an odd, memoryless and monotonic saturating nonlinearity φ the describing function $N(a)$ is real, and that

$$0 = N(\infty) \leqslant N(a) \leqslant m = N(0),$$

where $m = \varphi'(0)$ (Vidyasagar 1993).

The solutions of (5) can be obtained graphically with a polar plot where $G(j\omega)$ and $-1/N(a)$ are represented. Points where both curves intersect each other give approximately the amplitude a and the frequency ω of a possible limit cycle. The Loeb criterion to check whether the limit cycle is stable or not is well known (Cook 1994). It should be remarked that the relevant intersections are for $\omega > 0$, while intersections with $\omega = 0$, not considered in the describing function method, correspond to the phenomenon already mentioned in § 2.1 of multiplicity of equilibrium points.

The transition from a non-intersecting situation to an intersecting situation is a meaningful measure of the possibility of the appearance of a limit cycle and is generally associated with the so-called Hopf bifurcation. Then, a Hopf bifurcation may take place when, moving the parameters, the shape of the polar plot of $G(j\omega)$ changes from not cutting $-1/N(a)$ to cutting it. For the class of nonlinearities considered in this paper, $\varphi'(0) = 1$, which assures that $N(0) = 1$. Therefore, a Hopf bifurcation is produced just when $G(j\omega)$ crosses the point $(-1, 0)$ on the complex plane, where $-1/N(a)$ initiates on the right, but this should imply, based on the Nyquist criterion, that the origin is no longer stable.

On the other hand, owing to the saturated nature of φ, $N(\infty) = 0$. This can lead to an interesting Hopf bifurcation that takes place when $G(j\omega)$ starts to cross $-1/N(a)$ not at the right boundary, as in the previous paragraph, but at the left boundary, at infinity, giving rise to a large limit cycle coming from infinity. This is the Hopf bifurcation B_∞ at infinity. Later, in § 3, it will be discussed more deeply for two-dimensional systems. All this is compatible with the stability of the origin and it has been studied by Glover (1989), Llibre and Ponce (1996, 1997) and Llibre and Sotomayor (1996). In that case, even if the operating point remains stable, that stability will no longer be global.

The approximate character of the first harmonic balance is well known. However, the error of the approximation can be analysed (Bergen *et al.* 1982) and any degree of precision can be reached using higher-order harmonics. In this last case the simplicity of the first harmonic balance is lost, but any degree of precision can be attained. In the sequel, only the first harmonic balance will be considered, but the possibility of using higher-order harmonics should not be rejected.

Summing up the previous results, it is easy to see that the visual inspection of $G(j\omega)$ gives clues to understanding the global behaviour of the system, and not only the local behaviour around the origin. The value of $G(0)$ will indicate the appearance of new equilibria. Moreover, points where $G(j\omega)$ crosses the part of the horizontal axis to the left of point $(-1, 0)$ for $\omega > 0$ generically indicate the existence of stable or unstable limit cycles.

In the next section, the generic case of a two-dimensional system will be fully analysed illustrating these ideas. It should be remarked that this method works for any finite dimension. When $n > 2$ the Loeb stability criterion must be used with care by taking into account the real part of the remaining $n - 2$ roots of the corresponding linear substitution problem (Libre and Ponce 1996).

3. Two-dimensional systems analysis

Consider a second-order system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

with a control law

$$u = -Kx = -[k_1 \quad k_2]x.$$

The closed-loop system is equivalent to an unity feedback linear system with the open-loop transfer function

$$G(s) = \frac{p(s)}{q(s)} = \frac{k_1 + k_2 s}{s^2 + a_1 s + a_2},$$

and then $G(0) = k_1/a_2$. The closed-loop characteristic polynomial is

$$p(s) + q(s) = s^2 + (a_1 + k_2)s + (a_2 + k_1).$$

It is assumed that the control law is chosen such that the closed-loop system is stable and satisfies some given specifications. The closed-loop stability conditions are $a_1 > -k_2$ and $a_2 > -k_1$.

Now normalized saturation is considered in the feedback path, to have the control system structure in figure 1. Then the closed-loop system can be written in the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varphi(Kx). \qquad (6)$$

There are four parameters (a_1, a_2, k_1, k_2) in the system, but the parameters k_1, and k_2, further than assuring the closed-loop stability, will be assumed positive (as usual) and fixed. Thus, the only parameters to be considered in the bifurcation analysis are a_1 and a_2.

In figure 3 the shapes of the polar plots of $G(\mathrm{j}\omega)$ in the parameter space (a_1, a_2) are displayed. To understand how these shapes change, it is convenient to start by computing possible intersections of $G(\mathrm{j}\omega)$ with the real axis. Writing

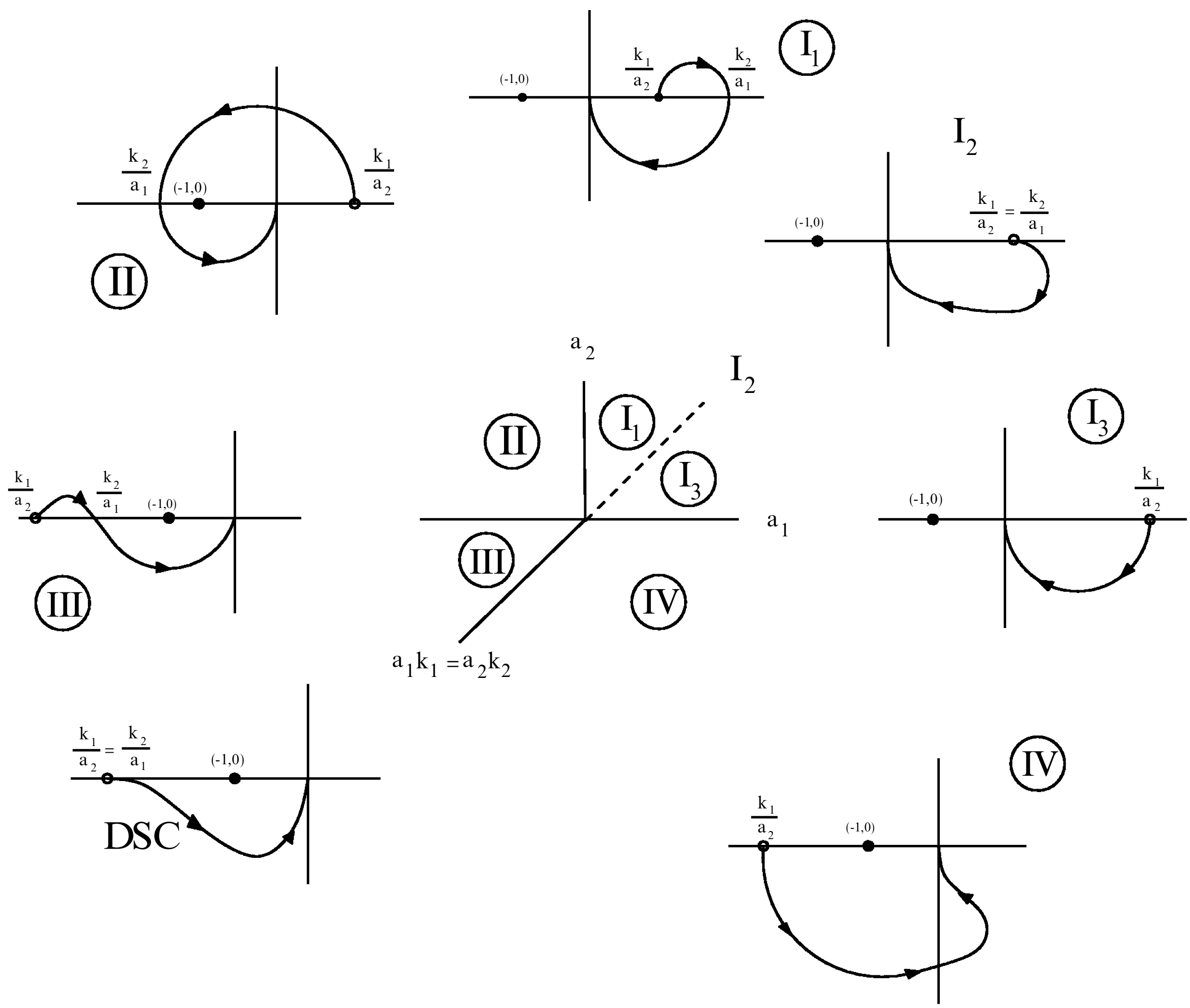

Figure 3. Shapes of the polar plots $G(\mathrm{j}\omega)$ in the parameter space (a_1, a_2).

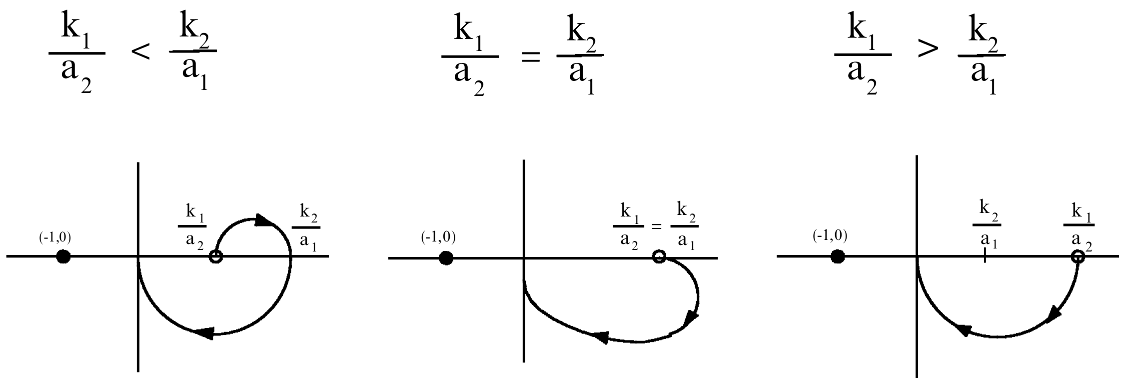

Figure 4. The polar plot of $G(j\omega)$ crosses the horizontal axis twice if the condition $k_2a_2 - k_1a_1 > 0$ is fulfilled, and once otherwise.

$$G(j\omega) = \frac{k_1 + j\omega k_2}{(a_2 - \omega^2) + j\omega a_1},$$

it is easy to deduce that, apart from the value $G(0) = k_1/a_2$, there will be another intersection with the horizontal axis if and only if

$$\frac{k_1}{a_2 - \omega^2} = \frac{k_2}{a_1},$$

and solving for ω^2 the condition $k_2a_2 - k_1a_1 > 0$ is obtained. Then, for $\omega^2 = a_2 - k_1(a_1/k_2)$, it follows that $G(j\omega) = k_2/a_1$ (figure 4).

Thus, the line $k_1a_1 = k_2a_2$ divides the diagram into two parts. According to the property just mentioned, in the upper part the polar plot of $G(j\omega)$ has, apart from $G(0)$, another crossing point with the real axis. Only in this half of the (a_1,a_2) plane could there be solutions with $\omega > 0$ to (5), and, therefore, limit cycles.

Figure 3 gives an overall picture of how the polar plot of $G(j\omega)$ evolves as parameters a_1 and a_2 are changed. According to the shape of $G(j\omega)$ there are four main regions denoted I, II, III and IV with different qualitative behaviours.

In region I (subdivided into regions I_1 and I_3), the system has a single attracting point, namely the origin, and is globally stable.

In region II, there is an unstable limit cycle around the stable origin.

In region III, there are two saddle points apart from the stable origin and the unstable limit cycle.

In region IV, there are the stable origin and two saddle points.

For this same case as in (6), Llibre and Sotomayor (1996) have obtained the whole bifurcation diagram using the qualitative theory of planar dynamical systems. A rather good approximation to the exact bifurcation diagram was obtained by Llibre and Ponce (1996) using arguments similar to those developed in this paper and it is shown in figure 5, where the state portrait of the four main regions is also included. In the boundary between regions I and II (the positive a_2 axis), a Hopf bifurcation B_∞ at infinity is produced. In the same way, the boundary between regions I and IV, and between regions II and III, is associated with a pitchfork bifurcation P_∞ at infinity (at the a_1 axis). Finally the frontier between regions III and IV is the line $k_1a_1 = k_2a_2$, where a double-saddle connection bifurcation (DSC) occurs, killing the unstable limit cycle. This line is not the exact boundary but only its approximation obtained by the first-order describing function techniques, and it constitutes the only quantitative discrepancy with the exact bifurcation diagram given by Llibre and Sotomayor (1996). In any case, the boundary exists and here a 'linearized' part of it has been found.

It should be emphasized that region I is the only region where the system is both open-loop stable and closed-loop globally stable. This leads to a concrete illustration of the comments in the introduction regarding the global stability problems of unstable plants. In regions II, III and IV, where the plant is unstable, the closed-loop system is only locally stable (provided that k_1 and k_2 have been chosen to assure the stability of the origin, as is currently assumed

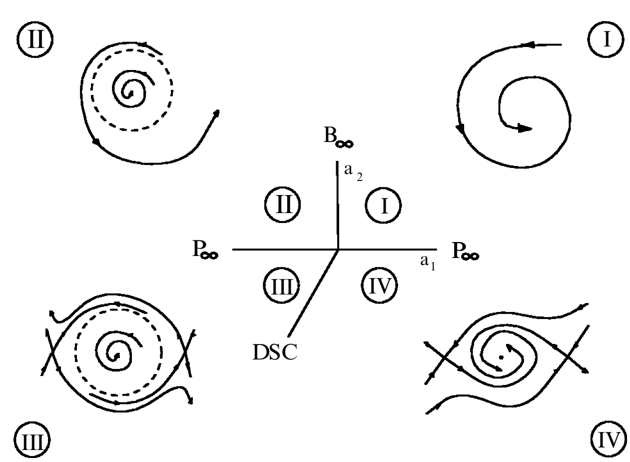

Figure 5. Bifurcation diagram, showing qualitatively the different state portraits for the four regions I, II, III and IV.

here). Obviously, this is because the system runs in an open-loop manner when the controller saturates.

The bifurcation diagram in figure 5 can be reinterpreted with the help of the polar plots of $G(\mathrm{j}\omega)$ in each of the regions as shown in figure 3. To that end an anticlockwise round trip starting from region I is proposed. This region is subdivided by the broken line I_2 ($k_1 a_1 = k_2 a_2$) in regions I_1 and I_3, with two different shapes for $G(\mathrm{j}\omega)$, but in both cases giving rise to polar plots that guarantee the global stability of the closed-loop system (neither multiple equilibria nor limit cycles can exist).

Entering from region I into region II through the Hopf bifurcation B_∞ at infinity causes the appearance of an unstable limit cycle. This bifurcation is easily reinterpreted with the polar plots. To enter region II, the value a_1 must be decreased. As shown above, the point where $G(\mathrm{j}\omega)$ crosses the real axis for $\omega \neq 0$ is k_2/a_1. Then, as a_1 tends to 0, this crossing point tends to ∞. For small values of $a_1 > 0$, it is easy to see that $G(\mathrm{j}\omega)$ has the shape shown in figure 6, where k_2/a_1 is the point where $G(\mathrm{j}\omega)$ cuts the horizontal axis for some $\omega > 0$. As a_1 tends to zero, the value of this intersection point grows tending to infinity, but, as a_1 reaches the value zero and becomes negative, then the shape of $G(\mathrm{j}\omega)$ of figure 6 changes to that in figure 7, k_2/a_1 being now negative. So, as a_1 crosses 0 and becomes negative, the crossing point will reappear from $-\infty$. However, taking a negative value, even if very large, means that the crossing point is on the real axis cutting $-1/N(a)$, and then giving rise to a limit cycle, which turns out to be unstable. This is the Hopf bifurcation B_∞ at infinity. Figure 8 shows the corresponding bifurcation diagram. For a complete description of this bifurcation see Glover (1989) and Llibre and Ponce (1997). Once in region II, the shape of $G(\mathrm{j}\omega)$ is consistent with the state portrait showing an unstable limit cycle.

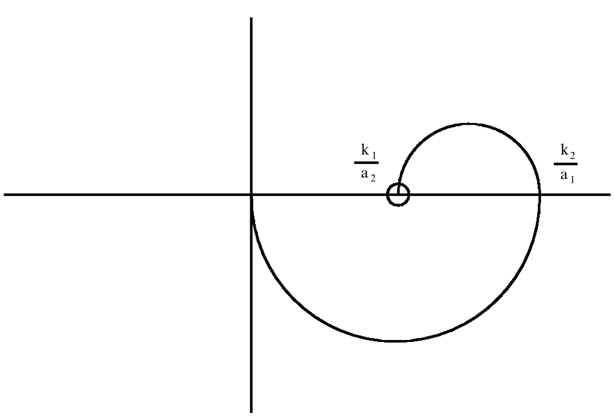

Figure 6. Polar plot of $G(\mathrm{j}\omega)$ for a_1 positive small.

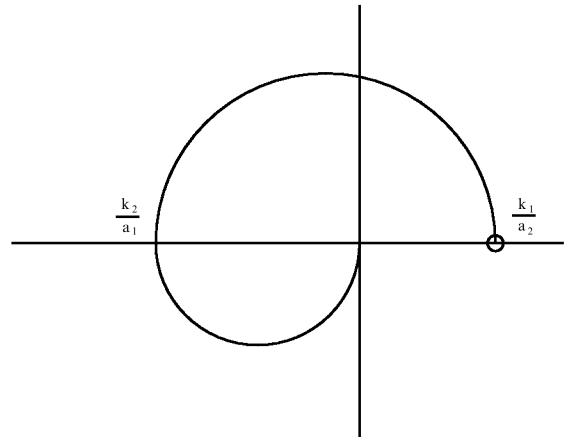

Figure 7. Polar plot of $G(\mathrm{j}\omega)$ for a_1 negative small.

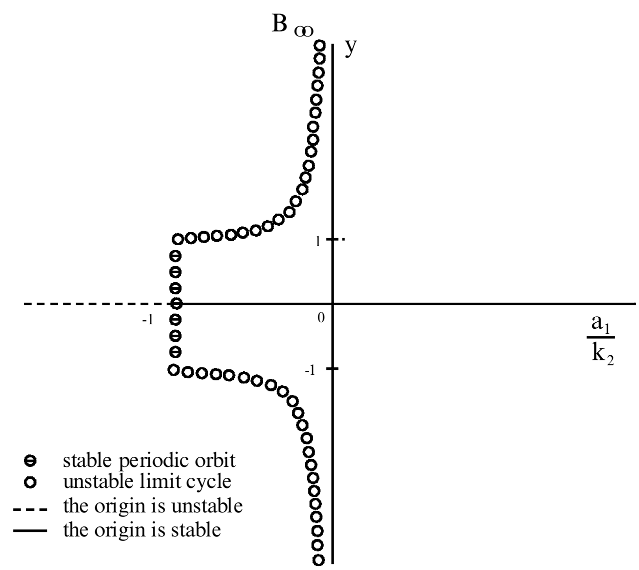

Figure 8. Bifurcation diagram as a function of a_1/k_2 when $k_2 a_2 - k_1 a_1 > 0$.

If one continues the journey anticlockwise, then region III is entered. For that a_2 must be decreased and $G(0) = k_1/a_2$ will tend to ∞. As a_2 reaches 0 and becomes negative, $G(0)$ comes from $-\infty$. But $-\infty < G(0) < -1$ means that two unstable equilibria will appear. Then for $a_2 = 0$ a pitchfork bifurcation P_∞ at infinity is produced. Figure 9 shows the corresponding bifurcation diagram.

Now, to complete the diagram, it remains to discuss what happens at line $k_1 a_1 = k_2 a_2$ in the third quadrant. On this line, as stated above, the morphology of the polar plot is changed and the crossing point of the real axis k_2/a_1 disappears. Then the unstable limit cycle degenerates in one DSC, to disappear when region IV is reached, where there are stable equilibria at the origin and two saddle points. This DSC bifurcation is easily interpreted with the polar plots as the coalescence of the two points where $G(\mathrm{j}\omega)$ cuts the horizontal axis.

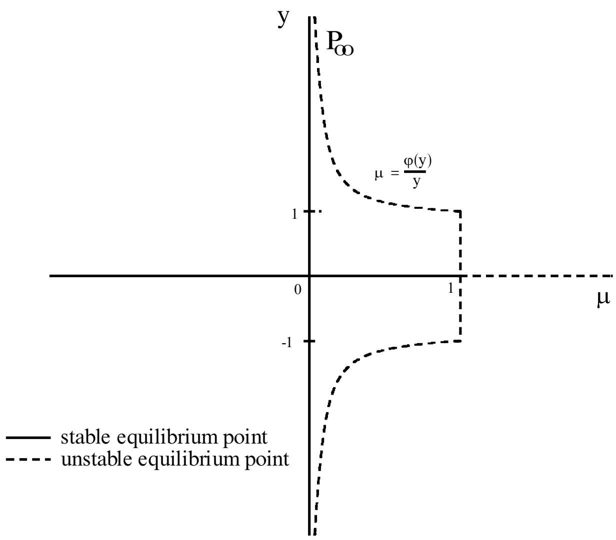

Figure 9. Bifurcation diagram as a function of $\mu = -1/G(0) = -a_2/k_1$.

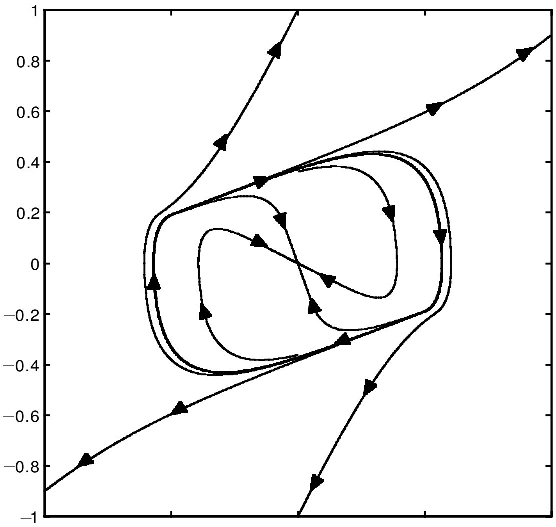

Figure 10. State portrait for the two-dimensional example. The unstable periodic orbit defines the basin of attraction of the origin.

Following the trip, one returns to region I through the pitchfork bifurcation P_∞ at infinity where the two saddle points disappear, leaving the state portrait with a single equilibrium point at the origin.

It should be recalled that the methodology applied here is based on the first-order harmonic balance and then is only approximated. However, the main qualitative aspects are retained and its application is straightforward for the engineer designing control systems with classical tools.

This kind of analysis is also relevant for conventional control design of systems showing windup problems (Seron *et al.* 1995, Middleton 1996, Ponce *et al.* 1999). Related research has been developed for systems with a delay (Pagano *et al.* 1996, 1999).

The above analysis is relevant to understand the effect of perturbations that change the initial conditions of the system, as is shown in the following example 1.

Example 1: Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

A linear control law of the form $u = -Kx$ has been designed to minimize the criterion $J = \int_0^\infty (x_1^2 + x_2^2 + u^2)\,dt$, obtaining $k_1 = 0.4142$ and $k_2 = 6.2907$. Suppose now additionally that the actuator saturates and the control signal is bounded. Then the origin is locally stable but, as $a_1 = -3$ and $a_2 = 1$, the system is in region II, which implies the existence of an unstable orbit around the origin, which gives rise to a limited attraction basin (figure 10). In this case, this basin is sufficiently small that a small perturbation, applied at the input, could make the system out of control as can be observed in figure 11. Curve a shows how, in spite of a perturbation of amplitude 0.5 applied at $t = 1$ and maintained for 0.75 s, the system again reaches the origin. Curve b corresponds to a greater perturbation of amplitude 1 of the same period, which now makes the system out of control. Regardless of the perturbation applied, the system goes out of control when the state vector does not belong to the attraction basin once the perturbation disappears. □

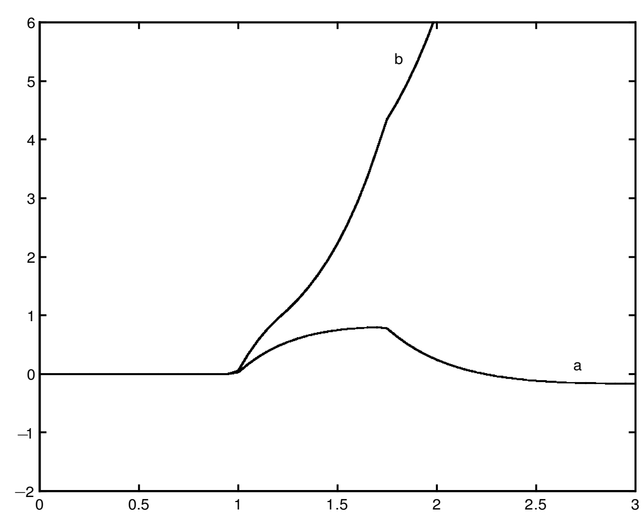

Figure 11. Response of the system to a perturbation. Curve a corresponds to a perturbation of amplitude 0.5 applied at $t = 1$ and maintained for 0.75 s. Curve b corresponds to a perturbation of amplitude 1 of the same period.

The results reported above for the case of normalized saturation can be easily generalized for a larger class of nonlinear systems. For instance, Aracil *et al.* (1998b) have found the same pattern of behaviours as in figure 5 for control systems where the plant is second-order linear and the controller is fuzzy.

The generalization of the analysis for higher-order linear systems is also straightforward, even if for such cases there is not a diagram as simple and synthetic as figure 5 (Llibre and Ponce 1996). However, the boundaries between regions of different qualitative behaviours are associated with bifurcations P_∞ and B_∞, like those previously mentioned. In the next section a three-dimensional will be considered.

4. Application to a three-dimensional system

Let us suppose that the open-loop transfer function is given by

$$G(s) = \frac{k_1 + k_2 s}{(s^2 + a_1 s + a_2)(s + a)}$$

and that there is normalized saturation in the feedback path, that is the system has the structure of figure 1.

The parameters k_1 and k_2 can be considered the proportional and derivative constants of the proportional–derivative (PD) controller designed to control the open-loop system. The pole at $s = -a$ will be assumed to be stable ($a > 0$).

4.1. *Stability analysis*

From the application of the Hurwitz criterion, the necessary and sufficient conditions for the stability of the linearized closed-loop system are

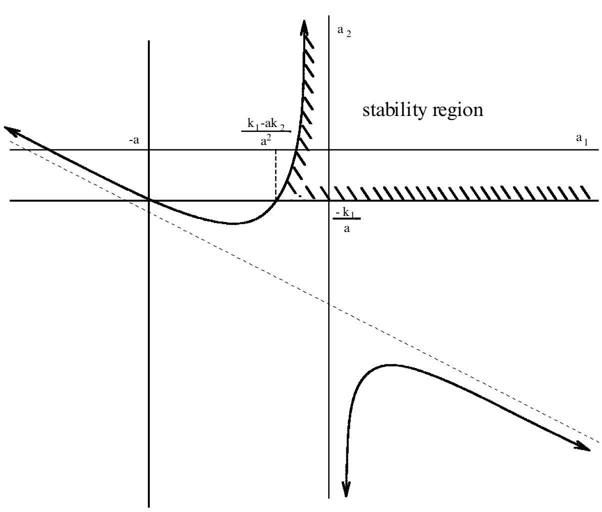

Figure 12. Region of stability in the (a_1, a_2) plane.

$$H_1(a_1, a_2) = a_1 + a > 0,$$
$$H_2(a_1, a_2) = (a + a_1)(a_1 a + a_2 + k_2) - (a_2 a + k_1) > 0,$$
$$H_3(a_1, a_2) = a_2 a + k_1 > 0.$$

As shown in figure 12, the horizontal line $a_2 = -k_1/a$ and one of the branches corresponding to $H_2(a_1, a_2) = 0$ form the frontier of the parameter stability region. Note that the curve $H_2(a_1, a_2) = 0$ intersects the aforementioned horizontal line at $a_1 = -a$ and at $a_1 = (k_1 - ak_2)/a^2$. Therefore, the size of the parameter stability region increases with increasing k_1 and $ak_2 - k_1$ and thus it is reasonable to suppose that in a practical control design both quantities are positive. Thus, in the following k_1 and $ak_2 - k_1$ will be assumed to be greater than zero.

In short, if $a > 0$ and the PD controller stabilizes the system then the assumptions $k_1 > 0$, $ak_2 - k_1 > 0$ and $a + a_1 > 0$ will be satisfied in most control designs. Also, the value of k_2 will be assumed to be less than a^2 as typically occurs when a is large enough (for instance, when it is associated with a high-frequency component). In what follows, the bifurcation diagram under these general assumptions will be obtained using the same ideas already employed for the two-dimensional case.

4.2. *Equilibrium points*

In this case, $G(0)$ is equal to $k_1/a_2 a$; so there will be more than one equilibrium point if $k_1/a_2 a < -1$. Under the assumption of closed-loop stability, this occurs only for negative a_2. Therefore, at $a_2 = 0$ there is a change in the number of equilibria, which in fact corresponds to a pitchfork bifurcation at the infinity.

4.3. *Solution of the harmonic balance equation*

Some algebraic manipulations show that, in the parameter stability region and under the above-mentioned condition $k_2 < a^2$, the polar plot of $G(j\omega)$ crosses the real axis at the frequency

$$\omega_c^2 = \frac{k_2 a_2 a - k_1(a_2 + a_1 a)}{k_2(a_1 + a) - k_1}.$$

The denominator $k_2(a_1 + a) - k_1$ is positive for all the regions of stability provided that the hypothesis $k_2 < a^2$ is satisfied. Effectively, the minimum value of this denominator is reached at the vertex of the stability region, where $a_1 = (k_1 - k_2 a)/a^2$. For that value of a_1, one obtains

$$k_2(a_1 + a) - k_1 = k_2\left(\frac{k_1 - ak_2}{a^2} + a\right) - k_1$$
$$= (ak_2 - k_1)\left(1 - \frac{k_2}{a^2}\right) > 0.$$

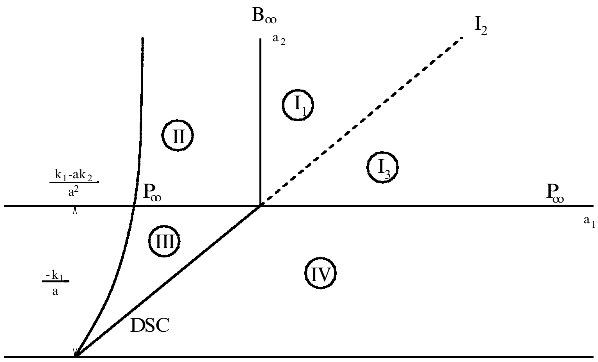

Figure 13. Bifurcation diagram for the region of stability in the (a_1, a_2) plane.

Thus, the line $k_2 a_2 a - k_1(a_2 + a_1 a) = 0$ divides the stability region into two parts. Note that this line contains the origin $(a_1, a_2) = (0, 0)$, and the vertex of the stability region $(a_1, a_2) = ((k_1 - ak_2)/a^2, -k_1/a)$ (figure 13). In the region that lies above the line, the polar plot crosses the real axis. In the region under the line, there is no frequency $\omega > 0$ such that the polar plots intersects the real axis.

As was seen for the two-dimensional system, $a_1 = 0$ corresponds to a Hopf bifurcation at infinity. Therefore, the axis $(a_1 = 0)$ divides the stability region that is over the line $k_2 a_2 a - k_1(a_2 + a_1 a) = 0$ into two regions. In the region placed to the left of the vertical axis and over the line, the polar plot of $G(j\omega)$ crosses the real axis at the left of the point $(-1, 0)$, which according to the describing function method may reveal the existence of a limit cycle.

In figure 13, it is shown how the stability region is divided into the four regions I, II, III and IV, at which polar plots of $G(j\omega)$ exhibit the same morphology as in regions I, II, III and IV of the two-dimensional system (figure 5). This is due to the above natural parameter restrictions, which assure that there exists at most one intersection between the polar plot of $G(j\omega)$ and the real axis, for $\omega > 0$.

5. Conclusions

A global perspective on the behaviour modes of control systems with saturation has been presented. This global perspective has been reached mainly using graphical frequency-domain tools. It has been shown how, depending on the polar plot of the open-loop transfer function $G(j\omega)$, the existence and stability of multiple equilibrium points and of limit cycles can be studied.

In isolation, these are well-known results, but here the bifurcation analysis framework has helped to integrate them into a unified conceptual scheme. Bifurcations have been shown to give rise to the boundaries of parameter regions with different qualitative behaviours. The full morphology of polar plots has been displayed for two-dimensional systems and a classification of the different behaviour modes of the nonlinear system has been reached. The same morphology has been found for a three-dimensional system with some reasonable assumptions on the parameter values. All this has been done using mainly frequency domain plots of the open-loop system, as in classical control methods.

Some attention has been addressed to the case of control of unstable plants with saturating controllers. It has been shown that even simple unstable plants with a saturating controller can give rise to global problems. These problems appear because the controller is designed to work well at the operating point, but for nonlinear systems this stability is only guaranteed around this point. Then the system is locally stable around the operating point but is not globally stable, so that large enough perturbations can lead it out of control. The moral is that in dealing with nonlinear systems one cannot avoid to tackling with global problems. This stresses the interest of global analysis of nonlinear control systems.

Acknowledgements

The authors appreciate the comments of E. Freire, D. Pagano and F. Gordillo and the recommendations of an anonymous referee.

This work has been partially supported by the Spanish Ministry of Education and Culture under grant CICYT TAP 97-0553.

References

Abed, E. H., Wang H. O., and Tesi, A., 1996, Control of bifurcations and chaos. *The Control Handbook,* edited by W. S. Levine (New York), pp. 951–966.

Alvarez, J., and Curiel, E., 1997, Bifurcations and chaos in a linear control system with saturated input. *International Journal Bifurcation and Chaos*. **7,** 1811–1821.

Alvarez, J., Curiel, E., and Verduzco, F., 1997, Complex dynamics in classical control systems. *Systems and Control Letters,* **31,** 277–285.

Aracil, J., Åström, K., and Pagano, D., 1998a, Global bifurcations in the Furuta pendulum. *Proceedings of NOLCOS, Enschede, The Netherlands, 1998, pp. 35–40.*

Aracil, J., Gordillo, F., and Álamo, T., 1998b, Global stability analysis of second-order fuzzy control systems. *Advances in Fuzzy Control,* edited by R. Palm, D. Driankov and H. Hellendorn (Berlin: Springer-Verlag), pp. 11–31.

Basso, M., Genesio, R., and Tesi, A., 1997, A frequency method for predicting limit cycle bifurcations. *Nonlinear Dynamics,* **13,** 339–360.

Bergen, A. R., Chua L. O., Mees, A. I., and Szeto, E. W., 1982, Error bounds for general describing function problems. *IEEE Transactions on Circuits and Systems,* **29,** 345–354.

Cook, P. A., 1994, *Nonlinear Dynamical Systems,* second edition (Englewood Cliffs, NJ: Prentice Hall).

Fukuma, A., Matsubara, M., Watanabe N., and Onogi, K., 1984, Bifurcations in the frequency response of nonlinear control systems. *IEEE Transactions in Automatic Control,* **29,** 450–452.

Genesio, R., and Tesi, A., 1992, Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems. *Automatica,* **28,** 531–548.

Glover, J. N., 1989, Hopf bifurcations at infinity. *Nonlinear Analysis,* **13,** 1393–1398.

Guckenheimer, J., and Holmes, P., 1983, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* (New York: Springer-Verlag).

Hale J., and Koçak, H., 1991, *Dynamics and Bifurcations* (Berlin: Springer-Verlag).

Kuznetsov, Y. A., 1995, *Elements of Applied Bifurcation Theory* (Berlin: Springer-Verlag).

Khalil, H. K., 1996, *Nonlinear Systems,* second edition (Englewood Cliffs, NJ: Prentice-Hall).

Kuznetsov, Y. A., 1995, *Elements of Applied Bifurcation Theory* (Berlin: Springer-Verlag).

Leonov, G. A., Ponomarenko, D. V., and Smirnova, V. B., 1996, *Frequency Methods for Nonlinear Analysis. Theory and Applications* (Singapore: World Scientific).

Llibre, J., and Ponce, E., 1996, Global first harmonic bifurcation diagram for odd piecewise linear control systems. *Dynamics and Stability of Systems,* **11,** 49–88; 1997, Hopf bifurcation from infinity for planar control systems. *Publicacions Matemàtiques,* **41,** 181–198.

Llibre, J., and Sotomayor, J., 1996, Phase portraits of planar control systems. *Nonlinear Analysis, Theory, Methods and Applications,* **27,** 1177–1197.

Mees, A. I., 1981, *Dynamics of Feedback Systems* (New York: Wiley).

Mehra, R. K., Kessel, W. C., and Carroll, J. V., 1977, Global stability and control analysis of aircraft at high angles-of-attack. Report ONRCR-215-248-1, US Office of Naval Research, Arlington, Virginia.

Middleton, R. H., 1996, Dealing with actuator saturation. The Control Handbook, edited by W. S. Levine (New York: IEEE), pp. 377–381.

Moiola, J., and Chen, G., 1993, Frequency domain approach to computation and analysis of bifurcations and limit cycles: a tutorial. *International Journal of Bifurcation and Chaos,* **3,** 843–867; 1996, *Hopf Bifurcation Analysis: a Frequency Domain Approach* (Singapore: World Scientific).

Nayfeh, A. H., and Balachandran, B., 1995, *Applied Nonlinear Dynamics* (New York: Wiley).

Pagano, D., Ponce, E., and Aracil, J., 1997, Bifurcations analysis of low-order nonlinear control systems with a delay. *Proceedings of the International Conference on Control of Oscillations and Chaos,* Vol. 5, Saint Petersburg, Russia, 27–29 August 1997, 478–483; 1999, Bifurcation analysis of time delay control systems with saturation. *International Journal of Bifurcation and Chaos,* **9,** 1089–1110.

Ponce, E., Alamo, T., and Aracil, J., 1996, Robustness and bifurcations for a class of piecewise linear control systems. *Proceedings of the 1996 IFAC World Congress,* Vol. E, paper 2b-02 5, 61–66.

Ponce, E., Aracil, J., and Pagano, D., 1999, Control systems with actuator saturation and bifurcations at infinity. *Proceedings of the Seventh Mediterranean Conference on Control and Automation,* Haifa, Israel, 1999, 1598–1608.

Seron, M. M., Goodwin, G. C., and Graebe, S. F.,1995. Control system design issues for unstable linear systems with saturated inputs. *Proceedings of the Institution of Electrical Engineers.* Pt D, **142,** 335–344.

Stein, 1989, Respect the unstable. *Proceedings of the 28th IEEE Conference on Decision and Control,* Tampa, Florida, USA, 1989.

Strogatz, S. H., 1995, *Nonlinear Dynamics and Chaos* (Reading, MA: Addison-Wesley).

Suàrez, R., Alvarez-Ramirez, J., and Aguirre, B., 1996, First harmonic analysis of planar linear systems with single saturated feedback. *International Journal of Bifurcation and Chaos,* **6,** 2605–2610.

Tarbouriech, S., and Burgat, Ch., 1997, Comments on the paper 'Some new results on closed-loop stability in the presence of control saturation'. *International Journal of Control,* **68,** 1202–1207.

Tesi, A., Abed, E. H., Genesio, R., and Wang, H. O., 1996, Harmonic balance analysis of period-doubling bifurcations with implications for control of nonlinear dynamics. *Automatica,* **32,** 1255–1271.

Vidyasagar, M., 1993, *Nonlinear Systems Analysis*, second edition (Englewood Cliffs, NJ: Prentice-Hall).